From estimate (23) and Eq.(9) it immediately follows that $p_1(0) = 0$. But then for some fairly small number $\delta > 0$ the inequality $\varphi(\sigma) + p_1(\sigma)\sigma < 0,\ \forall\sigma \in [0, \delta]$ holds. Hence, from inequality (22) and the estimate $p_1(\sigma) < p_0(\sigma)$ it follows that for fairly small δ the following inequality holds:

$$K_2^2(\sigma - \delta) + \mu K_2(\sigma - \delta) + \varphi(\sigma) + p_1(\sigma)\sigma < 0,\ \forall\ \sigma \in (\delta, a_2)$$

This means that $q = K_2(\sigma - \delta)$ is a non-contact straight line for Eq.(8) when $\sigma \in (\delta, a_2)$. But then $\alpha_2 \geqslant \delta > 0$.

Thus, for the conditions of Theorem 2 to hold, it is sufficient that inequality (21) holds.

For Lorenz's system, written in the standard form (12), this condition will take the form

$$\frac{(\sigma_1 + 1)^2}{(r-1)\sigma_1} > 4\frac{\sigma_1 + \Lambda}{\sigma_1 - \Lambda}, \quad \Lambda = \frac{2\sigma_1 - b}{2 + b(\sigma_1 + 1)\sigma_1^{-1}(r-1)^{-1}} \tag{24}$$

It is obvious that condition (24) holds when $\sigma_1 = 10,\ b = {}^8/_3,\ r \leqslant 2$. For large values of σ_1, r estimate (24) takes the form $r < {}^1/_4(b\sigma_1)^{1/2}$.

REFERENCES

1. SHIL'NIKOV L.P., Theory of bifurcations and Lorenz's model. MARSDEN J. and MacKRACKEN D., Bifurcation of cycle production and its applications. Moscow: Mir, 1980.
2. BARBASHIN E.A. and TABUEVA V.A., Dynamic systems with a cylindrical phase space. Moscow: Nauka, 1969.
3. BELYKH V.N. and NEKORKIN V.I., The qualitative analysis of multidimensional phase systems. Sib. matem. zh., 4, 18, 1977.
4. BELYKH V.N., Qualitative methods of the theory of non-linear oscillations of lumped systems. Uchebnoe posobie. Gor'kii: Izd-vo Gor'k, un-ta, 1980.
5. LEONOV G., TSCHSCHIJOWA T. and REITMANN V., Eine Frequenzvariante der Vergleichmethode von Belych - Nekorkin in der Theorie der Phasensynchronisation. Wiss. Z. Techn. Univ., Dresden, 32, 1, 1983.
6. LEONOV G.A., The method for the non-local reduction in the theory of absolute stability of non-linear systems. Avtomatika i telemekhanika, 2, 1984.
7. LEONOV G.A., The global statbility of Lorenz's system. PMM, 47, 5, 1983.
8. SPARROW C., The Lorenz equations: bifurcations, chaos and strange attractors. N.Y. Springer, 1982.

Translated by H.Z.

PMM U.S.S.R.,Vol.49,No.5,pp. 663-666,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON THE SINGULARITY OF THE STRESSES NEAR THE FACE OF A THIN ELASTIC INCLUSION IN A PIECEWISE HOMOGENEOUS PLANE*

D.V. GRILITSKII, A.A. EVTUSHENKO and YU.I. SOROKATYI

The asymptotic behaviour of normal stresses near the tip of a thin elastic inclusion situated near a line weld joining two dissimilar elastic half-planes, is studied. It is established that apart from the well-known root-type singularity /1/ two additional terms of the asymptotic expression exist which must not be neglected. One of them is of the order of unity, and the other contains an "imaginary singularity" and makes a singificant contribution to the state of stress when the distances between the face of the inclusion and the line separating the materials are small.

1. Normal stresses and their asymptotic behaviour. A thin elastic inclusion of normalized length 2 (here and henceforth all distances will be expressed in terms of the half-length of the inclusion), is situated in one of the welded isotropic half-planes possessing different elastic characteristics. The distance between the right end of the inclusion and the line separating the materials is δ (Fig.1). A field of tensile stresses σ_1 and σ_2 exists at a sufficient distance from the inclusion, and we have $\sigma_2 = \sigma_1(1 + \varkappa_1)\mu_2/[(1 + \varkappa_2)\mu_1]$, $\varkappa_j = (3 - \nu_j)/(1 + \nu_j)$ for the generalized plane stress state, $\varkappa_j = 3 - 4\nu_j$ for plane stress, μ_j is the shear modulus and ν_j is Poisson's ratio of the materials of the half-planes $(j = 1, 2)$. The corresponding quantities with zero index refer to the material of the inclusion.

**Prikl.Matem.Mekhan.*,49,5,864-867,1985

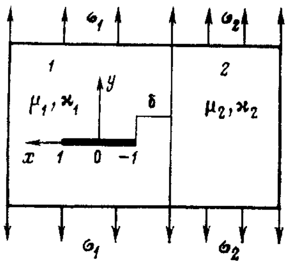

Fig.1

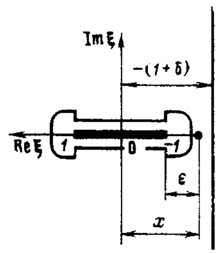

Fig.2

Taking into account the small thickness of the inclusion, we can model it by a jump in the values of the tangential stresses σ_{1xy} and normal displacements u_{1y} on the line $y = 0$:

$$\sigma_{1xy}(x) = -{}^1/_2 f_1(x), \quad \partial u_{1y}/\partial x = -{}^1/_2 f_2(x), \quad -1 \leqslant x \leqslant 1 \tag{1.1}$$

Assuming that the transverse deformations at the inclusion edges are equal, we obtain the following two conditions of interaction of a thin elastic inclusion with the matrix:

$$\partial u_{1x}/\partial x = k_0\sigma_x - k_1\sigma_{1y}, \quad u_{1y}/h = k_0\sigma_{1y} - k_1\sigma_x \tag{1.2}$$

$$\sigma_x = N_{-1} - \frac{1}{2h}\int_{-1}^{x}\sigma_{1xy}(t)\,dt, \quad k_0 = \frac{1+\varkappa_0}{8\mu_0}, \quad k_1 = \frac{3-\varkappa_0}{8\mu_0}$$

where N_{-1} is the normal stress at the end $x = -1$ of the inclusion and h is its relative thickness. Conditions (1.2) enable us to obtain solutions of problems for inclusions of arbitrary rigidity, ranging from perfect rigid $(\mu_0 = \infty)$ to perfectly pliable $(\mu_0 = 0)$, modelling a slit.

The problem formulated above was solved in /2/ using the integral Mellin transform and interaction conditions (1.2). A system of integral equations with singular, Cauchy-type kernel is written for the functions of the jump (1.1) sought. A numerical solution of this system in the class of functions possessing an integrable singularity at the points $x = \pm 1$. is constructed using the method of mechanical quadratures and Gauss-Chebyshev nodes. Knowing the functions $f_1(x)$ and $f_2(x)$, we obtain the normal stresses σ_{1y} on the continuation of the axial line of the inclusion $(-1 < x < -1-\delta)$ using the formulas /2/

$$\sigma_{1y}(x) = -\frac{1-\varkappa_1}{2(1+\varkappa_1)}\frac{1}{\pi}\int_{-1}^{1}\frac{F_1(t)\,dt}{t-x} + \frac{2\mu_1}{1+\varkappa_1}\frac{1}{\pi}\int_{-1}^{1}\frac{F_2(t)\,dt}{t-x} + \tag{1.3}$$

$$\frac{1}{2\pi}\int_{-1}^{1}\sum_{j=1}^{2}K_{1j}(x,t)F_j(t)\,dt, \quad m = \frac{\mu_2}{\mu_1}, \quad F_j(t) = \frac{f_j(t)}{\sqrt{1-t^2}}$$

$$K_{1j}(x,t) = \sum_{k=0}^{2}c_{kj}(x+1+\delta)^k\frac{d^k}{dx^k}(t+x+2+2\delta)^{-1} \quad (j = 1, 2)$$

$$\bar{c}_{k1} = \bar{c}_{k1}/(1+\varkappa_1), \; c_{k2} = -\bar{c}_{k2}\cdot 2\mu_1/(1+\varkappa_1) \; (k = 0, 1, 2)$$
$$\bar{c}_{01} = m_5 + 3(2-\varkappa_1)m_6, \; \bar{c}_{11} = 2(7+\varkappa_1)m_6, \; \bar{c}_{21} = 4m_6$$
$$\bar{c}_{02} = m_5 + 3m_6, \; \bar{c}_{12} = 12m_6, \; \bar{c}_{22} = 4m_6, \; m_1 = m\varkappa_1 - \varkappa_2$$
$$m_2 = m + \varkappa_2, \; m_3 = 1 - m, \; m_4 = 1 + m\varkappa_1, \; m_5 = m_1/m_2, \; m_6 = m_3/m_4$$

Integrating along the contour L in the complex $\xi = x + iy$ plane (Fig.2) and determining the residues at the point $\xi = x$ we obtain, from (1.3),

$$\sigma_{1y}(x) = -\frac{1-\varkappa_1}{2(1+\varkappa_1)}F_1(x) + \frac{2\mu_1}{1+\varkappa_1}F_2(x) + \sum_{j=1}^{2}\sum_{k=0}^{2}(-1)^k c_{kj}(x + \tag{1.4}$$

$$1+\delta)^k\varphi_{kj}(x_1) + \frac{C_0}{2}$$

$$\varphi_{0j}(x_1) = \frac{F_j(x_1)}{2}, \quad \varphi_{1j}(x_1) = \frac{1}{2}\left[\frac{f_j'(x_1)}{\sqrt{x_1^2-1}} - \frac{x_1 f_j(x_1)}{(\sqrt{x_1^2-1})^3}\right]$$

$$\varphi_{2j}(x_1) = \frac{1}{2}\left[\frac{f_j''(x_1)}{\sqrt{x_1^2-1}} - \frac{x_1 f_j'(x_1)}{(\sqrt{x_1^2-1})^3} + \frac{(1+2x_1^2)f_j(x_1)}{(\sqrt{x_1^2-1})^5}\right],$$

$$x_1 = -(x+2+2\delta)$$

The right-hand side of relation (1.4) contains four singularities. Two of them $x = \pm 1$ correspond to the usual root-type singularity at the ends of a thin elastic inclusion /1/. The other pair of points $x = -1-2\delta$ and $x = -3-2\delta$, represents the image of the points $x = \pm 1$ reflected in the line separating the materials of the half-planes. We shall call them "imaginary", since expression (1.4) holds for $x \geqslant -1-\delta$. However, subsequent numerical analysis has shown that the imaginary singularity $x = -1-2\delta$ makes a significant contribution to the stress σ_{1y} at fairly small values of δ, i.e. when the right tip of the inclusion is sufficiently near the line separating the materials.

Using the results of /2/, we obtain the following expressions for the normal stresses in the second half-plane:

$$\sigma_{2y} = \int_{-1}^{1} \sum_{j=1}^{2} K_{2j}(x, t) F_j(t)\, dt$$

$$K_{21}(x, t) = -\frac{m}{2\pi m_2 m_4} K(\varkappa_1), \quad K_{22}(x, t) = \frac{\mu_2}{\pi m_2 m_4} K(-1)$$

$$K(\varkappa_1) = m_4 \frac{3t - x + 2 + 2\delta}{(t-x)^2} - m_2 \frac{(2+\varkappa_1)\, t - \varkappa_1 x + 2 + 2\delta}{(t-x)^2}$$

Using contour integration we can show that

$$\sigma_{2y} = -\frac{m}{2m_2 m_4} \left\{ \frac{f_1(x)}{\sqrt{x^2-1}} \left\{ m_4 \left[3 - \frac{2x(x+1+\delta)}{x^2-1} \right] - m_2 \left[2 + \varkappa_1 - \frac{2x(x+1+\delta)}{x^2-1} \right] \right\} + 2(x+1+\delta)(m_4 - m_2) \frac{f_1'(x)}{\sqrt{x^2-1}} \right\} + \frac{\mu_2}{m_2 m_4} \left\{ \frac{f_2(x)}{\sqrt{x^2-1}} \left\{ m_4 \left[3 - \frac{2x(x+1+\delta)}{x^2-1} \right] - m_2 \left[1 - \frac{2x(x+1+\delta)}{x^2-1} \right] \right\} + 2(x+1+\delta)(m_4 - m_2) \frac{f_2'(x)}{\sqrt{x^2-1}} \right\} + \frac{C_0'}{2} \tag{1.5}$$

Here, as before, C_0' is the result of contour integration along a circle of radius R as $R \to \infty$. We note that solution (1.5) can be used when $x \leqslant -1 - \delta$.

Let us investigate the asymptotic behaviour of the normal stresses. Using the substitution $x = -1 - \varepsilon$ we transfer the origin of coordinates to the tip $x = -1$ of the inclusion and consider the corresponding expressions for σ_{1y} and σ_{2y} when $\varepsilon \ll 1$, remembering that δ is of the order of ε. We introduce the stress intensity coefficients in such a manner, that well-known results are obtained in the case of a crack or of a perfectly rigid inclusion.

Such coefficients at the tip $x = -1$ will be

$$k_1^1 = -\frac{1-\varkappa_1}{2(1+\varkappa_1)} \frac{f_1(-1)}{\sqrt{a_0}}, \quad k_1^2 = \frac{2\mu_1}{1+\varkappa_1} \frac{f_2(-1)}{\sqrt{a_0}}$$

(a_0 is the half-length of the inclusion). Then expression (1.4) can be written in the following form:

$$\sigma_{1y} = \sum_{j=1}^{2} \frac{k_1^j \sqrt{a_0}}{\sqrt{2}} \left[\frac{1}{\sqrt{\varepsilon}} - \beta_j \sum_{k=0}^{2} \frac{(-1)^k \alpha_k \varepsilon^{kj}}{2\sqrt{2\delta - \varepsilon}} \left(\frac{\delta - \varepsilon}{2\delta - \varepsilon} \right)^k \right] + \frac{C_0}{2} + O(\sqrt{\varepsilon}) \tag{1.6}$$

$$\alpha_0 = 1,\ \alpha_1 = 0.5,\ \alpha_2 = 0.75,\ \beta_1 = 2/(1+\varkappa_1),\ \beta_2 = 1$$

Relation (1.6) contains, apart from the known root-type singularity, the "imaginary" singularity mentioned above, whose contribution may be more significant for small values of δ, then that of the terms of the order of $\varepsilon^{-1/2}$.

To study the stresses in the second material, we will consider Eq.(1.5) for $\varepsilon > \delta$, $\delta = O(1)$. Then we have

$$\sigma_{2y} = \frac{(1+\varkappa_1)\, m}{(1-\varkappa_1)\, m_2 m_4} \left\{ m_4 \left[3 - \frac{\varepsilon - \delta}{\varepsilon} \right] - m_2 \left[2 + \varkappa_1 - \frac{\varepsilon - \delta}{\varepsilon} \right] \right\} \frac{k_1^1 \sqrt{a_0}}{\sqrt{2\varepsilon}} + \frac{(1+\varkappa_1)\, m}{2m_2 m_4} \left\{ m_4 \left[3 - \frac{\varepsilon - \delta}{\varepsilon} \right] - m_2 \left[1 - \frac{\varepsilon - \delta}{\varepsilon} \right] \right\} \frac{k_1^2 \sqrt{a_0}}{\sqrt{2\varepsilon}} + \frac{C_0'}{2} + O(\sqrt{\varepsilon}) \tag{1.7}$$

2. Analysis of the computational results.

Fig.3 shows the behaviour of normal stresses $\sigma_{jy}\ (j = 1, 2)$ near the tip of the inclusion situated near the line separating the materials of the half-planes made of aluminium and an epoxy resin ($m = 23.08$; $\nu_1 = 0.35$; $\nu_2 = 0.3$) for $\delta = 0.0005$. The relative rigidity of the inclusion $k = \mu_0'/\mu_1$ is equal to 10^{-3} (the solid lines), 10 (the dashed lines) and 10^3 (the dot-dash lines). The dimensionless width of the inclusion is $h = 0.01$. The curves marked 1, 2, 3 correspond to three different computational schemes: 1 depicts the exact solution obtained using Eq.(1.1) and the results of /2/, 2 depicts the solution obtained using the usual asymptotic representation taking into account only the singularity of the type $\varepsilon^{-1/2}$, when the stress intensity coefficients are known, and 3 depicts the solution obtained with help of the formulas (1.6), (1.7) with $C_0 = C_0' = 0$ and the same values of the stress intensity coefficients.

In all cases, comparison with the exact solution (curves 1) shows that the new asymptotic solution (curves 3) yields a correct qualitative pattern of the distribution of normal stresses and differs only by a constant, which is obviously caused by the choice of $C_0 = C_0' = 0$. Curves 2 differ noticeably from the exact solution both qualitatively and quantitatively (the disappearance of the minimum for more pliable inclusions $k < 1$, and the change of sign for more rigid inclusions $k > 1$).

Fig.4 gives the results of calculations for an inclusion situated in a material with greater shear modulus ($m = 0.0433$; $\nu_1 = 0.3$; $\nu_2 = 0.35$; $k = 0.1$). In this case the normal stresses σ_{2y} are, as expected, vanishingly small and can be neglected.

Let us indicate how the results obtained can be used in practice. When the stress intensity coefficients are obtained from interferometric data, the stresses σ near the tip of an acute-angled defect can be written in the form /3/

$$\sigma = \sum_{n=0}^{N} A_n \varepsilon^{n-1/2} + C \tag{2.1}$$

where A_n ($n = 0, 1, \ldots N$), C are unknown constants and $A_0 = (k_1^1 + k_1^2)\sqrt{a_0/2}$. For small distances $\varepsilon < \varepsilon_0$ we can put $N = 0$, $C = 0$. However, as a rule, a zone of premature disintegration forms in this region, i.e. the stresses relax, the material becomes brittle and behaves non-linearly. Moreover, optical caustic curves appear which obscure the region of measurements. In distant regions $\varepsilon > \varepsilon_1$ the choice of N becomes difficult, and some of the asymptotic terms included may be smaller than the accuracy of the measurements. The two-parameter representations ($N = 0$, $C \neq 0$) are used in most cases. The region of reliability $\varepsilon_0 < \varepsilon < \varepsilon_1$ is chosen by establishing the correlations between the measureable quantities σ and $\varepsilon^{-1/2}$.

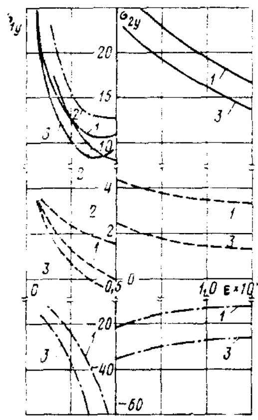

Fig.3

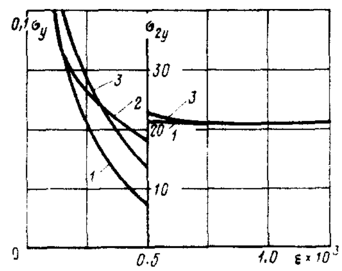

Fig.4

The above investigations show that near the boundary separating the materials we must put $C = A + B/\sqrt{2\delta - \varepsilon}$, and this improves the quality of the asymptotic expression (2.1) considerably. The parameters A_0, A, B are found from the known values of σ at a sufficient number of points obtained from the experimental data or from numerical solutions.

REFERENCES

1. DRAGAN S.M. and OPANASOVICH V.K., State of stress of a strip (beam) with a rectilinear thin-walled inclusion. PMM 43, 2, 1979.
2. EVTUSHENKO A.A., Elastic equilibrium of a composite plane containing an arbitrarily oriented thin elastic inclusion. PMM 44, 5, 1980.
3. SCHROEDL M.A., McGOWAN J.J. and SMITH C.W., Determination of stress-intensity factors from photoelastic data with applications to surface-flow problems. - Exp. Mech. 14, 10, 1974.

Translated by L.K.